Student Name: ______________________

Teacher: ___________________________

Sydney Technical High School

2024 | HIGHER SCHOOL CERTIFICATE **TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for careless work or illegible writing
- **Begin each question on a new page**

Total marks: 100

Section I — 10 marks

- Attempt Questions 1 - 10
- Allow about 15 minutes for this part

Section II — 90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 15 minutes for this section.

SECTION I

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Answer each question on the multiple-choice page in your answer booklet.

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1 Which one of the following is i^{2019} simplified?

A. i

B. $-i$

C. 1

D. -1

2 Consider the expansion of

$$(11x - 8)^{11} = \sum_{k=0}^{11} t_k x^k.$$

Which expression represents t_k?

A. $t_k = {}^{11}C_k(11x)^k(-8)^{11-k}$

B. $t_k = {}^{11}C_k(11)^{11-k}(-8)^k$

C. $t_k = {}^{11}C_k(11)^k(-8)^{11-k}$

D. $t_k = {}^{11}C_k(11x)^k(-8)^k$

3 A particle undergoing simple harmonic motion in a straight line has an acceleration given by $\ddot{x} = 75 - 25x$, where x is the displacement after t seconds. Where is the centre of the motion?

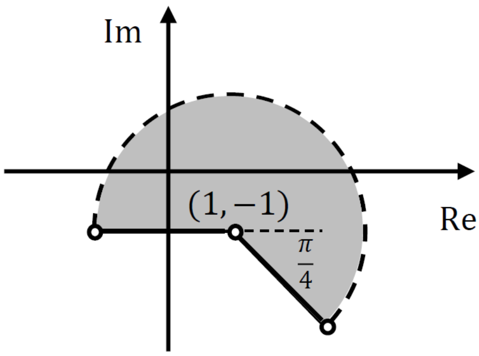
A. $x = 0$

B. $x = 3$

C. $x = 5$

D. $x = 6$

4 The shaded region below is constructed by taking the intersections of two other regions.



Which of the following best represents the two possible regions?

A. $|z - 1 + i| < 2$ and $-\frac{\pi}{4} \leq arg(z - 1 + i) \leq \pi$

B. $|z - 1 + i| \leq 2$ and $-\frac{\pi}{4} < arg(z - 1 + i) < \pi$

C. $|z + 1 - i| < 2$ and $-\frac{\pi}{4} \leq arg(z + 1 - i) \leq \pi$

D. $|z + 1 - i| \leq 2$ and $-\frac{\pi}{4} < arg(z + 1 - i) < \pi$

5 Which of the following lines is parallel to $x = 3 + 2t, y = 2 - t, z = 3 + 4t$?

A. $\frac{x-2}{5} = \frac{y-1}{2} = \frac{z-5}{3}$

B. $\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-5}{-1}$

C. $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-5}{4}$

D. $\frac{x-2}{5} = \frac{y-1}{1} = \frac{z-5}{7}$

6 It is given that $2 - i$ is a root of $P(z) = z^3 + az^2 + bz + 20$ where a and b are real numbers. Factorize $P(z)$ over the real numbers.

A. $P(z) = (z + 4)(z^2 + 4z + 5)$

B. $P(z) = (z + 4)(z^2 - 4z - 5)$

C. $P(z) = (z + 4)(z^2 - 4z + 5)$

D. $P(z) = (z - 4)(z^2 - 4z + 5)$

7 Consider the following statement.

"If it is raining, then there are grey clouds in the sky."

Which of the following is the contrapositive of this statement?

A. If there are grey clouds in the sky, then it is raining.

B. If it is not raining, then there are no grey clouds in the sky.

C. If it is raining, then there are no grey clouds in the sky.

D. If there are no grey clouds in the sky, then it is not raining.

8 At time t, the position vector of a particle is given by $\underset{\sim}{r}(t) = \sqrt{t}\,\underset{\sim}{i} + \frac{t-1}{t}\,\underset{\sim}{j}$. With what equation does this particle moves along the path?

A. $y(x^2 + 1) = 1$

B. $x(y^2 - 1) = 1$

C. $y = x^2 - 1$

D. $y = \frac{x^2-1}{x^2}$

9 Consider the following definite integrals

$$I_1 = \int_2^1 \frac{1}{\sqrt{4 - x^2}} \ dx$$

$$I_2 = \int_0^1 \cos(\cos^{-1} x) \, dx$$

$$I_3 = \int_0^1 \sin^{-1} x \ \ dx$$

$$I_4 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos(x^2) \cos(2x^2) \quad dx$$

Which of the following correctly lists the values of these definite integrals in

ascending order?

A. I_1, I_2, I_3, I_4

B. I_4, I_2, I_3, I_1

C. I_1, I_4, I_3, I_2

D. I_1, I_4, I_2, I_3

10 $f(x)$ is any continuous and differentiable functions over $\mathbb{R}$.

Given $2 + 3f(x) = f(-x) + 3\int_{-1}^{1} f(x)\, dx$, what is the value of $\int_{-1}^{1} f(x)\, dx$?

A. 2

B. 1

C. -1

D. -2

SECTION II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours and 45 minutes for this section

Answer each question in your answer booklet. Start each question on a new page.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) START A NEW PAGE

(a) If $z = 3 - i$ and $w = 1 + 2i$, find in the form $a + ib$, where a and b are real, the values of

(i) $z - 2w$ **1**

(ii) $z\overline{w}$ **1**

(iii) $\frac{z}{w}$ **1**

(b) Find $\int \frac{x}{1+x^2}\,dx$ **1**

(c) (i) Express $-2 + 2i$ in exponential form. **2**

(ii) Hence, simplify $(-2 + 2i)^{8k}$, where k is an integer. **2**

Question 11 continues on page 9

Question 11 (continued)

(d) A particle is experiencing simple harmonic motion along a straight line. At time t seconds, its displacement x metres from a fixed point O on the line is given by

$$x = 8\cos^2 t + 1$$

(i) Show that $\ddot{x} = -4(x-5)$. **2**

(ii) Find the centre and period of motion. **2**

(e) Evaluate $\int_3^4 x\sqrt{x-3}\,dx$. **3**

End of Question 11

Question 12 (16 marks) START A NEW PAGE

(a) Find the size of the angle between the vectors $\begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$, correct to the nearest degree. **2**

(b) Prove the point $\begin{pmatrix} 8 \\ 12 \\ -16 \end{pmatrix}$ does not lie on $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ **2**

(c) (i) Split $\frac{8}{(x+2)(x^2+4)}$ into partial fractions. **2**

(ii) Hence, evaluate $\int_0^2 \frac{8}{(x+2)(x^2+4)} dx$. **3**

(d) Find the intersections of the line $\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$ with the sphere $(x-3)^2 + (y-2)^2 + (z+1)^2 = 36$. **3**

(e) On the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral. **2**

(ii) Express $z_2 - z_1$ in *exact* modulus-argument form. **2**

End of Question 12

Question 13 (14 marks) START A NEW PAGE

(a) A particle is moving in simple harmonic motion in a straight line and its velocity is given by $v^2 = 6 - x - x^2$ m/s, where x is the displacement in metres.

(i) Find the two points between which the particle is oscillating. **1**

(ii) Find the centre of motion. **1**

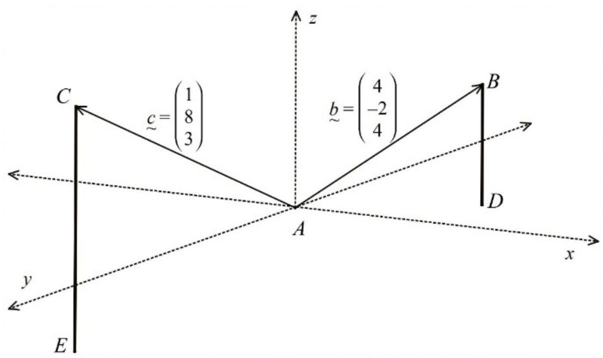
(iii) Find the maximum speed of the particle. **1**

(iv) Find the acceleration of the particle in terms of x. **1**

(b) Use the substitution $t = \tan x$ to find $\int \frac{dx}{1+\sin 2x}$ **3**

(c) Two vertical posts CE and BD stand on a horizontal plane ADE. **2**

Using A as the origin, the vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ represent the locations of the top of each post, B and C.



Show that ΔABC is right angled at A.

Question 13 continues on page 12

Question 13 (continued)

(d) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$, where n is a non-negative number.

(i) Show that $I_n = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$, when $n \geq 2$. **2**

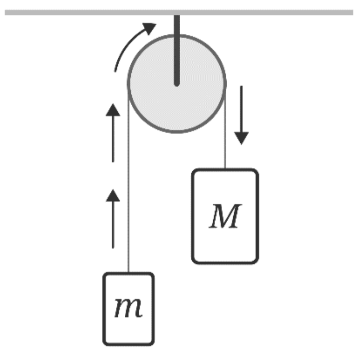
(ii) Deduce that $I_n = \frac{(n-1)}{n} I_{n-2}$, when $n \geq 2$. **2**

(iii) Evaluate I_4. **1**

End of Question 13

Question 14 (14 marks) START A NEW PAGE

(a) Box m has a mass of 4 kg and Box M has a mass of 6 kg. The boxes are connected by a light inextensible string passing over a frictionless pulley. Initially, the boxes are at rest. After Box m has travelled x metres in an upwards direction, it is travelling at v metres per second. **3**



Prove that $v = \sqrt{\frac{2gx}{5}}$

(b) Use mathematical induction to prove that **3**

$$\frac{d^n}{dx^n}(xe^{2x}) = 2^{n-1}(2x+n)e^{2x}$$

for $n \geq 1$.

Question 14 continues on page 14

Question 14 (continued)

(c) A mass of 1 kg is falling under gravity (g) through a medium in which the resistance to the motion is proportional to the square of the velocity.

Let k be the constant of proportionality.

(i) Write an equation for the acceleration of this mass. **1**

(ii) Show that the mass reaches a terminal velocity given by $v = \sqrt{\frac{g}{k}}$. **1**

(iii) Show that the distance it has fallen when it reaches a velocity v m/s is given by **3**

$$x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

(d) Prove by contradiction, or otherwise, that for $a \geq 2$; **3**

$$\sqrt{a} + \sqrt{a+2} > \sqrt{a+8}.$$

End of Question 14

Question 15 (15 marks) START A NEW PAGE

(a) (i) Show that $1, 2$ and 3 are the only positive integers that satisfy **1**

$$a^2 + b^2 + c^2 = 14$$

(ii) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ are the position vectors of the points at the ends of a diameter of a sphere.

(α) Find the centre and radius of the sphere. **2**

(β) Determine how many points, with integer coefficients, lie on the sphere (include the two given points). **2**

(b) (i) Prove that **2**

$$\int_{\frac{1}{a}}^{a} \frac{f(x)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)} dx = \int_{\frac{1}{a}}^{a} \frac{f\left(\frac{1}{x}\right)}{x\left(f(x) + f\left(\frac{1}{x}\right)\right)} dx$$

(ii) Hence evaluate **3**

$$I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x\left(\sin x + \sin \frac{1}{x}\right)} dx$$

Question 15 continues on page 16

Question 15 (continued)

(c) Let n be an integer greater than 2. Suppose ω is an nth root of unity and $\omega \neq 1$.

(i) By expanding the left-hand side, show that **2**

$$(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1) = n$$

(ii) Using the identity $\frac{1}{z^2-1} = \frac{z^{-1}}{z-z^{-1}}$, or otherwise, prove that **1**

$$\frac{1}{\cos 2\theta + i \sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$$

provided that $\sin\theta \neq 0$.

(iii) Hence, if $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, find the real part of $\frac{1}{\omega-1}$. **1**

(iv) Deduce that **1**

$$1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5} = -\frac{5}{2}$$

End of Question 15

Question 16 (16 marks) START A NEW PAGE

(a) α and β are the roots of $x^2 - 2x + 4 = 0$.

(i) Prove that **3**

$$\alpha^n - \beta^n = i2^{n+1}\sin\left(\frac{n\pi}{3}\right)$$

(ii) Hence, find the value of $\alpha^9 - \beta^9$. **1**

(b) (i) Use de Moivre's theorem to show that **3**

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(ii) Find the general solution of $\tan 4\theta = 1$. **1**

(iii) Hence find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in trigonometrical form. **2**

(c) It is known that the arithmetic mean of three numbers is greater than the geometric mean of those numbers, that is

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

(i) Given that $a + b + c = t$ and $(a, b, c > 0)$, prove that **3**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{t}$$

(ii) Hence, given $a + b + c = 1$ and $(a, b, c > 0)$, prove that **3**

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8$$

End of paper

STHS Ext 2 Trial Solutions.

① $i^{2019} = i^{2019-2016}$

$= i^3$

$= i^2 \times i$

$= -1 \times i$

$= -i$ B.

② $(11x-8)^{11} = \sum_{k=0}^{11} {}^{11}C_k \, (11x)^k (-8)^{11-k}$

$= \sum_{k=0}^{11} {}^{11}C_k \, 11^k (-8)^{11-k} \, x^k$

$t^k = {}^{11}C_k \, 11^k (-8)^{11-k}$ C.

③ $\ddot{x} = 75 - 25x$

$= -25(x-3)$

$= -5^2(x-3)$

$x = 3$ is centre of motion B.

④ Circle centre $(1,-1)$ and radius is 2.

$|z-(1-i)| < 2$

$|z-1+i| < 2$

$-\frac{\pi}{4} \le \arg(z-1+i) \le \pi$ A.

⑤
$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

↑

compare coefficients with denominators C.

(6) $P(z) = z^3 + az^2 + bz + 20$

Let roots be $2 \pm i$ and α.

$$(2+i)(2-i)\alpha = -\frac{20}{1}$$

$$(4-i^2)\alpha = -20$$

$$5\alpha = -20$$

$$\alpha = -4.$$

$$P(z) = (z - -4)(z-(2+i))(z-(2-i))$$

$$= (z+4)(z-2-i)(z-2+i)$$

$$= (z+4)(z^2-4z+5) \qquad C.$$

(7) Contrapositive for: If P then Q

is: If not Q then not P. D.

(8) $x = \sqrt{t}$ $\qquad y = \frac{t-1}{t}$

$x^2 = t$

$$y = \frac{x^2-1}{x^2} \qquad D.$$

(9) $I_1 < 0$ $\qquad I_3 = (\frac{\pi}{2} \times 1) - \int_0^{\frac{\pi}{2}} \sin x \, dx$ (using areas)

$I_2 = \int_0^1 x \, dx$ $\qquad = \frac{\pi}{2} - 1$

$= \frac{1}{2}$ $\qquad = 0.5707...$

$I_4 = 0$ (odd f^n).

I_1, I_4, I_2, I_3 $\qquad$ D.

(10) $2 + 3f(x) = f(-x) + 3\int_{-1}^{1} f(x)\,dx$.

Let $I = \int_{-1}^{1} f(x)\,dx$.

$2 + 3f(x) = f(-x) + 3I$.

$2\int_{-1}^{1} dx + 3\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx + 3I\int_{-1}^{1} dx$.

$2[x]_{-1}^{1} + 3I = \underbrace{\int_{-1}^{1} f(-x)\,dx}_{} + 3I[x]_{-1}^{1}$

$\rightarrow$ Let $u = -x$

$-du = dx$.

$x = -1,\ u = 1$

$x = 1,\ u = -1$

$4 + 3I = \int_{1}^{-1} -f(u)\,du + 6I$

$4 = \int_{-1}^{1} f(u)\,du + 3I$

$4 = I + 3I$

$I = 1$. B.

(11) a) $z = 3 - i$, $w = 1 + 2i$

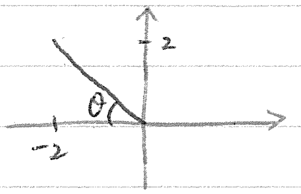
(i) $z - 2w = 3 - i - 2(1+2i)$
$= 3 - i - 2 - 4i$
$= 1 - 5i$

(ii) $z\bar{w} = (3-i)(1-2i)$
$= 3 - 7i + 2i^2$
$= 1 - 7i$

(iii) $\frac{z}{w} = \frac{3-i}{1+2i} \times \frac{1-2i}{1-2i}$
$= \frac{3 - 7i + 2i^2}{1 - 4i^2}$
$= \frac{1 - 7i}{5}$

b) $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$
$= \frac{1}{2}\ln|1+x^2| + C$

c) (i) $-2 + 2i$



$r = |-2+2i|$
$= \sqrt{2^2 + 2^2}$
$= 2\sqrt{2}$

$\theta = \pi - \tan^{-1}\left(\frac{2}{2}\right)$
$= \frac{3\pi}{4}$

$-2 + 2i = 2\sqrt{2}e^{i\frac{3\pi}{4}}$

(ii) $\left(2\sqrt{2}e^{\frac{3\pi i}{4}}\right)^{8k} = \left(2^{\frac{3}{2}}\right)^{8k} e^{\frac{3\pi i}{4} \times 8k}$
$= 2^{12k} e^{6k\pi i}$

(1.1) d) $x = 8\cos^2 t + 1$

(i) $x = 8\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) + 1$

$= 4 + 4\cos 2t + 1$

$= 4\cos 2t + 5 \rightarrow x - 5 = 4\cos 2t$

$\dot{x} = 4 \times 2 \times -\sin 2t$

$= -8\sin 2t$

$\ddot{x} = -8 \times 2\cos 2t$

$= -4 \times 4\cos 2t$

$= -4(x-5)$.

(ii) $c = 5$

$T = \frac{2\pi}{2}$

$= \pi$ seconds.

e) $\int_3^4 x\sqrt{x-3}\,dx$

Let $u = x - 3$

$du = dx$.

$= \int_0^1 (u+3)u^{\frac{1}{2}}\,du$.

$= \int_0^1 \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right) du$

$= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + 3\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1$

$= \frac{2}{5} + 3\left(\frac{2}{3}\right)$

$= \frac{12}{5}$

(12) a) $\cos\theta = \dfrac{5(-3)+3(2)+2(5)}{\sqrt{5^2+3^2+2^2}\times\sqrt{(-3)^2+2^2+5^2}}$

$= \dfrac{1}{38}$

$\theta = 88°$

b) $\begin{pmatrix} 8 \\ 12 \\ -16 \end{pmatrix} = \begin{pmatrix} -1+3\lambda \\ 0+4\lambda \\ 1+-5\lambda \end{pmatrix}$ $\quad 8=-1+3\lambda,\ \lambda=3$

$12 = 4\lambda,\ \lambda=3$

$-16 = 1-5\lambda,\ \lambda=\frac{17}{5}$

$\therefore$ the point does not lie on $\underset{\sim}{r}$.

c)(i) $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{a}{x+2} + \dfrac{bx+c}{x^2+4}$.

$8 = a(x^2+4)+(bx+c)(x+2)$

$8 = ax^2+4a+bx^2+cx+2bx+2c$

$8 = (a+b)x^2+(2b+c)x+(4a+2c)$.

$a+b=0$

$a=-b.$ $\quad -①$

$2b+c=0$

$2(-a)+c=0$ $\quad$ using ①.

$-2a+c=0$ $\quad -②$

$4a+2c=8$ $\quad -③$

$4a-2(2a)+2c-2(c)=8-2(0)$

$8a=8$

$a=1$

$b=-1$

$c=2.$

$\dfrac{8}{(x+2)(x^2+4)} = \dfrac{1}{x+2} + \dfrac{2-x}{x^2+4}.$

(12) c) (ii) $\int_0^2 \left(\frac{1}{x+2} + \frac{2-x}{x^2+4}\right) dx$.

$= \int_0^2 \frac{1}{x+2}\, dx + \int \frac{2}{x^2+4}\, dx - \frac{1}{2}\int \frac{2x}{x^2+4}\, dx$.

$= \left[\ln|x+2| + 2\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(x^2+4)\right]_0^2$

$= (\ln 4 + \tan^{-1} 1 - \frac{1}{2}\ln 8) - (\ln 2 + \tan^{-1} 0 - \frac{1}{2}\ln 4)$

$= \ln 4 + \frac{\pi}{4} - \frac{1}{2}\ln 8 - \ln 2 + \frac{1}{2}\ln 4$.

$= 2\ln 2 + \frac{\pi}{4} - \frac{3}{2}\ln 2 - \ln 2 + \ln 2$.

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$.

d). Parametric equations: $x = 3 + 2\lambda$

$y = 2 - \lambda$

$z = -1 - 2\lambda$.

$(x-3)^2 + (y-2)^2 + (z+1)^2 = 36$.

$(3+2\lambda-3)^2 + (2-\lambda-2)^2 + (-1-2\lambda+1)^2 = 36$

$4\lambda^2 + \lambda^2 + 4\lambda^2 = 36$

$9\lambda^2 = 36$

$\lambda^2 = 4$

$\lambda = \pm 2$.

$\lambda = -2$: $x = 3 - 4 = -1$

$y = 2 + 2 = 4$

$z = -1 + 4 = 3$ $\quad (-1, 4, 3)$.

$\lambda = 2$: $x = 3 + 4 = 7$

$y = 2 - 2 = 0$

$z = -1 - 4 = -5$ $\quad (7, 0, -5)$

Points of intersection are $(-1, 4, 3)$ and $(7, 0, -5)$.

(12) e) (i) $z_1 = 2\operatorname{cis}\frac{4\pi}{5}$, $z_2 = 2\operatorname{cis}\frac{7\pi}{15}$.

$|z_1| = 2$

$|z_2| = 2$

$= |z_1|$, i.e. $OA = OB$.

$\angle AOB = \arg z_1 - \arg z_2$

$= \frac{\pi}{3}$

$\angle ABO$ (opposite OA) and

$\angle BAO$ (opposite OB) are equal

since angles opposite equal sides

are equal.

$\angle ABO = \angle BAO$

$= (\pi - \angle AOB) \div 2$

$= \frac{\pi}{3}$.

$\therefore \angle ABO = \angle BAO = \angle AOB$ and

$\triangle OAB$ is equilateral.

(ii) $z_2 - z_1 = \overrightarrow{AB}$

$\overrightarrow{AB}$ is $\overrightarrow{OB}$ rotated by $\frac{\pi}{3}$ clockwise.

$z_2 - z_1 = z_2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$= 2\operatorname{cis}\left(\frac{7\pi}{15} - \frac{\pi}{3}\right)$

$= 2\operatorname{cis}\frac{2\pi}{15}$.

(13) (a) $v^2 = 6 - x - x^2$ m/s.

(i) $v = 0$, displacement is max.

$0 = 6 - x - x^2$

$0 = (3+x)(2-x)$

$x = -3, 2.$

Particle oscillates between -3m and 2m.

(ii) $\frac{-3+2}{2} = -\frac{1}{2}$ m.

(iii) max. speed @ $x = -\frac{1}{2}$ (centre).

$v^2 = 6 - (-\frac{1}{2}) - (-\frac{1}{2})^2$

$= \frac{25}{4}$

$v = \pm \frac{5}{2}$ m/s

Maximum speed is at $2{\cdot}5$ m/s.

(iv) $v^2 = 6 - x - x^2$

$\frac{1}{2}v^2 = 3 - \frac{x}{2} - \frac{x^2}{2}.$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right).$

$= \frac{d}{dx}\left(3 - \frac{x}{2} - \frac{x^2}{2}\right)$

$= -\frac{1}{2} - \frac{1}{2}(2x)$

$\therefore \ddot{x} = -\frac{1}{2} - x.$

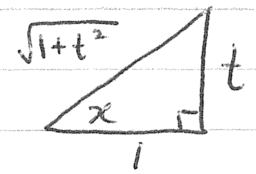
(13) (b) $\displaystyle\int \frac{dx}{1+\sin 2x}$

$t = \tan x$

$dt = \sec^2 x \, dx$

$= (1+t^2)\, dx$

$dx = \dfrac{dt}{1+t^2}$



$$= \int \frac{1}{1+2\sin x \cos x}\, dx$$

$$= \int \frac{1}{1+2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}} \frac{dt}{1+t^2}$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \frac{dt}{1+t^2}$$

$$= \int \frac{1+t^2}{1+t^2+2t} \frac{dt}{1+t^2}$$

$$= \int \frac{1}{(t+1)^2}\, dt$$

$$= \int (t+1)^{-2}\, dt$$

$$= \frac{(t+1)^{-1}}{-1} + C$$

$$= -\frac{1}{t+1} + C$$

$$= -\frac{1}{\tan x+1} + C$$

13 (c). $\angle ABC$ found using $\underset{\sim}{b} \cdot \underset{\sim}{c} = |\underset{\sim}{b}| \cdot |\underset{\sim}{c}| \cos\theta$.

If $\cos 90° = 0$, then $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$ and $\angle ABC = 90°$

$$\underset{\sim}{b} \cdot \underset{\sim}{c} = 1 \times 4 \times 8 \times (-2) + 3 \times 4$$
$$= 4 - 16 + 12$$
$$= 0.$$

$\therefore \angle ABC = 90°$ and $\triangle ABC$ is a right-angled triangle.

13 (d). $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$.

(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \times \sin x \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \times \frac{d}{dx}(-\cos x) \, dx$.

$= \left[\sin^{n-1} x \times (-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \times \frac{d}{dx}(\sin^{n-1} x) dx$

$= 0 + \int_0^{\frac{\pi}{2}} \cos x \left[(n-1)\sin^{n-2} x \cos x\right] dx$.

$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$.

(ii) $I_n = (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$.

$= (n-1) \int \sin^{n-2} x - \sin^n x \, dx$.

$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$= (n-1) I_{n-2} - (n-1) I_n$.

$I_n + (n-1) I_n = (n-1) I_{n-2}$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$.

(iii) $I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx$

$= \left[1\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

$I_2 = \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{4}$

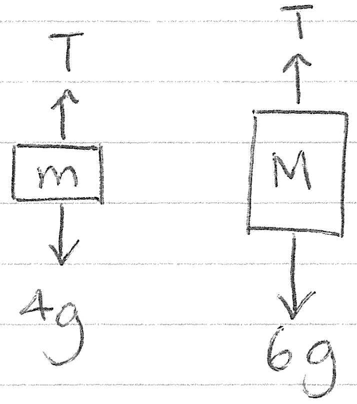
$I_4 = \frac{3}{4} \times \frac{\pi}{4}$

$= \frac{3\pi}{16}$.

(14) (a) $4\ddot{x} + 6\ddot{x} = (6g - T) - (4g - T)$

$$10\ddot{x} = 2g$$

$$\ddot{x} = \frac{g}{5}$$



$$v \cdot \frac{dv}{dx} = \frac{g}{5}$$

$$\frac{dv}{dx} = \frac{g}{5v}$$

$$\frac{dx}{dv} = \frac{5}{g}v$$

$$x = \frac{5}{g}\int_0^v v\,dv$$

$$= \frac{5}{g}\left[\frac{v^2}{2}\right]_0^v$$

$$x = \frac{5v^2}{2g}$$

$$\frac{2gx}{5} = v^2$$

$$v = \pm\sqrt{\frac{2gx}{5}}$$

$v = \sqrt{\frac{2gx}{5}}$ only as Box m is moving upwards.

(14) (b) Let $P(n)$ represent the proposition.

Show $P(n)$ true for $n=1$, i.e.

$$\frac{d}{dx}(xe^{2x}) = 2^{1-1}(2x+1)e^{2x}.$$

$$LHS = \frac{d}{dx}(xe^{2x}) \qquad RHS = 2^0(2x+1)e^{2x}$$
$$\qquad\qquad\qquad\qquad = (2x+1)e^{2x}$$
$$= e^{2x} + x \cdot 2e^{2x} \qquad = LHS$$
$$= (2x+1)e^{2x}$$

$\therefore$ True for $n=1$.

Assume $P(n)$ true for $n=k$, i.e.

$$\frac{d^k}{dx^k}(xe^{2x}) = 2^{k-1}(2x+k)e^{2x}.$$

Show true for $n=k+1$, i.e.

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{2x}) = 2^{k+1-1}(2x+k+1)e^{2x}$$
$$= 2^k(2x+k+1)e^{2x}.$$

$$LHS = \frac{d^{k+1}}{dx^{k+1}}(xe^{2x})$$
$$= \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^{2x})\right)$$
$$= \frac{d}{dx}\left(2^{k-1}(2x+k)e^{2x}\right) \quad \text{using assumption.}$$
$$= 2^{k-1} \times \left[(2x+k) \times 2e^{2x} + 2 \times e^{2x}\right]$$
$$= 2^{k-1} \times 2e^{2x}(2x+k+1)$$
$$= 2^{k-1} \times 2 \times (2x+k+1) \times e^{2x}$$
$$= 2^{k-1+1}(2x+k+1)e^{2x}$$
$$= 2^k(2x+k+1)e^{2x}$$
$$= RHS$$

(14)

(c) (i) $\ddot{x} = g - kv^2$

(ii) When $\ddot{x} = 0$, $g = kv^2$

$$v^2 = \frac{g}{k}$$

$$v = \sqrt{\frac{g}{k}}.$$

(iii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - kv^2$$

$$\frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}.$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = -\frac{1}{2k}\int \frac{-2kv}{g - kv^2}\,dv$$

$$= -\frac{1}{2k}\ln(g - kv^2) + C$$

$x = 0, v = 0$

$$0 = -\frac{1}{2k}\ln(g) + C$$

$$C = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln(g - kv^2) + \frac{1}{2k}\ln g$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

(14)(d) Assume $\sqrt{a} + \sqrt{a+2} \leq \sqrt{a+8}$

$$(\sqrt{a} + \sqrt{a+2})^2 \leq a+8.$$
$$a + 2\sqrt{a}\sqrt{a+2} + a+2 \leq a+8.$$
$$2a + 2\sqrt{a(a+2)} + 2 \leq a+8$$
$$a + 2\sqrt{a(a+2)} \leq 6.$$

When $a=2$, $2+2+2\sqrt{2(2+2)} > 6$.

$2+2\sqrt{2(2+2)} < 3+2\sqrt{3(3+2)}$

$\therefore$ assumption is false.

15 (a) (i) $1^2+2^2+3^2=1+4+9$

$=14$

Also note that $4^2>14$, so a, b and c must be less than 4.

(ii) (α)

centre $=\begin{pmatrix}2\\2\\2\end{pmatrix}$ and radius $=\sqrt{14}$.

(β) We require distance from centre to the coordinates to be $\sqrt{14}$ and of the form

$$\begin{pmatrix}2\pm a\\2\pm b\\2\pm c\end{pmatrix}$$

where $a^2+b^2+c^2=14$, satisfying conditions from part (i).

The permutations of $\pm1, \pm2, \pm3$ is $\underbrace{3!}_{\text{is the permutations of } 1,2,3.}\times\underbrace{2\times2\times2}_{\text{each ordinate can be positive or negative.}}=48.$

⑮ (b) (i) $\text{LHS} = \int_{\frac{1}{a}}^{a} \frac{f(x)}{x(f(x)+f(\frac{1}{x}))}\,dx.$ $\quad u = \frac{1}{x} \quad \frac{du}{dx} = -\frac{1}{x^2}$

$$= -u^2$$

$$dx = -\frac{du}{u^2}$$

$$= \int_{a}^{\frac{1}{a}} \frac{f(\frac{1}{u})}{\frac{1}{u}\left[f(\frac{1}{u})+f(u)\right]} \times \left(-\frac{du}{u^2}\right)$$

$$= \int_{\frac{1}{a}}^{a} \frac{f(\frac{1}{u})}{u\left[f(u)+f(\frac{1}{u})\right]}\,du.$$

$$= \int_{\frac{1}{a}}^{a} \frac{f(\frac{1}{x})}{x\left[f(x)+f(\frac{1}{x})\right]}\,dx$$

$= \text{RHS}$

(ii)

$$I = \int_{\frac{1}{2}}^{2} \frac{\sin x}{x(\sin x + \sin\frac{1}{x})}\,dx.$$

$$= \int_{\frac{1}{2}}^{2} \frac{\sin(\frac{1}{x})}{x(\sin x + \sin\frac{1}{x})}\,dx.$$

$$2I = \int_{\frac{1}{2}}^{2} \frac{\sin x + \sin\frac{1}{x}}{x(\sin x + \sin\frac{1}{x})}\,dx.$$

$$I = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{x}\,dx.$$

$$= \frac{1}{2}\left[\ln x\right]_{\frac{1}{2}}^{2}$$

$$= \frac{1}{2}\left(\ln 2 - \ln\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\ln 2 + \ln 2\right)$$

$$= \ln 2$$

Q15 (c)(i) $w^n = 1$, $1 + w + w^2 + w^3 + \dots + w^{n-1} = 0$

$(1 + 2w + 3w^2 + 4w^3 + \dots + nw^{n-1})(w - 1)$

$= w + 2w^2 + 3w^3 + 4w^4 + \dots + (n-1)w^{n-1} + nw^n$
$\quad -1 - 2w - 3w^2 - 4w^3 - 5w^4 - \dots - nw^{n-1}$

$= -1 - w - w^2 - w^3 - w^4 - \dots - w^{n-1} + nw^n$

$= -(1 + w + w^2 + w^3 + w^4 + \dots + w^{n-1}) + n$

$= n \qquad$ (since $1 + w + w^2 + w^3 + \dots + w^{n-1} = 0$

(ii) Let $z = \text{cis}\,\theta$
$= \cos\theta + i\sin\theta$
$\bar{z} = \cos\theta - i\sin\theta$

$z - \bar{z} = 2i\sin\theta$

$z^2 - 1 = \text{cis}\,2\theta - 1$
$= \cos 2\theta + i\sin 2\theta - 1$

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{1}{z^2 - 1}$$

$$= \frac{z^{-1}}{z - z^{-1}}$$

$$= \frac{\bar{z}}{z - \bar{z}} \quad \text{since } |z| = 1$$

$$= \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$$

If $\sin\theta = 1$, then

$$\cos 2\theta + i\sin 2\theta - 1$$
$$= 1 - 2\sin^2\theta + 2i\sin\theta\cos\theta - 1$$
$$= \sin\theta(-2\sin\theta + 2i\cos\theta)$$
$$= 0.$$

(iii) $$\frac{1}{w-1} = \frac{1}{\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} - 1}$$

$$= \frac{\cos\frac{\pi}{n} - i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}$$

$$= -\frac{1}{2} - \frac{i}{2}\cot\frac{\pi}{n}.$$

$$\therefore \operatorname{Re}\left(\frac{1}{w-1}\right) = -\frac{1}{2}.$$

(iv) $1 + 2w + 3w^2 + 4w^3 + \ldots + nw^{n-1} = \frac{n}{w-1}$

$n=5$: $1 + 2w + 3w^2 + 4w^3 + 5w^4 = \frac{5}{w-1}$ (1)

where $w = \operatorname{cis}\frac{2\pi}{5}$

$$\operatorname{Re}(w^r) = \operatorname{Re}\left[\operatorname{cis}\frac{2\pi}{5}\right]^r$$
$$= \operatorname{Re}\left(\cos\frac{2\pi r}{5} + i\sin\frac{2\pi r}{5}\right)$$
$$= \cos\frac{2\pi r}{5} \quad (2)$$

$$\operatorname{Re}(1 + 2w + 3w^2 + 4w^3 + 5w^4) = \operatorname{Re}\left(\frac{5}{w-1}\right)$$
$$1 + 2\operatorname{Re}(w) + 3\operatorname{Re}(w^2) + 4\operatorname{Re}(w^3) + 5\operatorname{Re}(w^4) = 5\operatorname{Re}\left(\frac{1}{w-1}\right)$$
$$1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5} = -5 \times \frac{1}{2}$$
$$= -\frac{5}{2}$$

(16) (a) $x^2 - 2x + 4 = 0$, α, β roots

(i) $x = \dfrac{2 \pm \sqrt{4-16}}{2}$

$= \dfrac{2 \pm 2\sqrt{3}i}{2}$

$= 1 \pm \sqrt{3}i$

$\alpha = 1+\sqrt{3}i$, $\beta = 1-\sqrt{3}i$

$|\alpha| = \sqrt{1+3}$
$= 2.$, $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$
$= \frac{\pi}{3}.$

$\alpha = 1+\sqrt{3}i$
$= 2\text{cis}\frac{\pi}{3}.$

$\alpha^n = 2^n \text{cis}\frac{n\pi}{3}.$

Similarly, $\beta^n = 1-\sqrt{3}i$
$= 2^n(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^n$
$= 2^n(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}).$

$\alpha^n - \beta^n = 2^n\cos\frac{n\pi}{3} + 2^n i\sin\frac{n\pi}{3}$
$- 2^n\cos\frac{n\pi}{3} + 2^n i\sin\frac{n\pi}{3}$
$= 2^n\left(2i\sin\frac{n\pi}{3}\right)$

$= 2^{n+1} i\sin\frac{n\pi}{3}.$

(ii) $\alpha^9 - \beta^n = 2^{10} i\sin\frac{9\pi}{3}$
$= 2^{10} i \sin 3\pi$
$= 0.$

(16) (b)

(i) $z = \text{cis}\,\theta$

$z^4 = \text{cis}\,4\theta$

$= \sum_{k=0}^{4} \binom{4}{k} i^k \sin^k\theta \cos^{4-k}\theta$

$\cos 4\theta = \binom{4}{0}\cos^4\theta + \binom{4}{2}(-\sin^2\theta)\cos^2\theta + \binom{4}{4}\sin^4\theta$

$\sin 4\theta = \binom{4}{1}\sin\theta\cos^3\theta + \binom{4}{3}(-\sin^3\theta)\cos\theta$

$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$

$= \dfrac{4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta}{\cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta}$ $\quad \div\cos^4\theta$ / $\div\cos^4\theta$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(ii) $\tan 4\theta = 1$

$4\theta = \tan^{-1}(1) + \pi n, \quad n \in J$

$= \dfrac{\pi}{4} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

$\theta = \dfrac{\pi}{16} + \dfrac{\pi n}{4}$

$= \dfrac{\pi(1+4n)}{16}, \quad n \in J.$

(iii) $\dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$ $\quad$ (let $\tan\theta = x$)

$4\tan\theta - 4\tan^3\theta = 1 - 6\tan^2\theta + \tan^4\theta$

$4x - 4x^3 = 1 - 6x^2 + x^4$

$x^4 + 4x^3 - 6x^2 + 4x^3 - 4x + 1 = 0$

$x = \tan\left[\frac{\pi}{16}(1+4n)\right], \quad n \in J$

$= \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{3\pi}{16}, -\tan\frac{7\pi}{16}$

(16) c)(i) Given $\frac{a+b+c}{3} \geqslant \sqrt[3]{abc}$

$$\frac{t}{3} \geqslant (abc)^{\frac{1}{3}}.$$

$$\frac{3}{t} \leq \frac{1}{(abc)^{\frac{1}{3}}}$$

$$\frac{1}{(abc)^{\frac{1}{3}}} \geqslant \frac{3}{t}. \quad \star.$$

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geqslant \sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geqslant 3\left(\frac{1}{abc}\right)^{\frac{1}{3}}.$$

$$\geqslant 3\left(\frac{3}{t}\right) \quad \text{using } \star$$

$$\geqslant \frac{9}{t}.$$

(ii) $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \left(\frac{1-a}{a}\right)\left(\frac{1-b}{b}\right)\left(\frac{1-c}{c}\right)$

$$= \frac{(1-a)(1-b)(1-c)}{abc}$$

$$= \frac{1-(a+b+c)+(ab+ac+bc)-abc}{abc}.$$

$$= \frac{\cancel{1-1}+ab+ac+bc-abc}{abc}$$

$$= \frac{1}{c}+\frac{1}{b}+\frac{1}{a}-1$$

$$\geqslant \frac{9}{t}-1$$

$$\geqslant \frac{9}{a+b+c}-1$$

$$\geqslant \frac{9}{1}-1$$

$$\geqslant 8.$$